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As R is a function of the coordinates and the time only, we can treat it as we have done r. Then

$$\frac{dR}{dL}\frac{dL}{dt}+\frac{dR}{dl}\left(\frac{dl}{dt}-n\right)+\frac{dR}{dG}\frac{dG}{dt}+\frac{dR}{dg}\frac{dg}{dt}+\frac{dR}{dH}\frac{dH}{dt}+\frac{dR}{dh}\frac{dh}{dt}=0.$$

On substituting in this the values of the differentials of the elements which have already been determined, it is seen that all the terms but two mutually cancel each other. And on dividing the result by $\frac{dR}{dg}$, we get

$$\frac{dg}{dt}=-\frac{dR}{dG}.$$

By adding to R the term $\frac{\mu^2}{2L^2}=\frac{\mu}{2a}$, its partial derivative with respect to L is augmented by the term $-\frac{\mu^2}{L^3}=-n$, but all the other derivatives are unchanged. In consequence of this addition the value of the differential of l becomes

$$\frac{dl}{dt}=-\frac{dR}{dL}.$$

An objection may be made against the preceding method of obtaining the differentials of l and g, in that the quantities $\frac{dr}{dl}$ and $\frac{dR}{dg}$, which both periodically vanish, have been employed as divisors. But this objection has force only when it is admitted that the differentials of l and g or the corresponding derivatives of R may be discontinuous. For having proved the truth of the equations for all times, except when the divisors, just mentioned, vanish, it follows, that if both members are continuous, the equations must still hold even for the moments of time when $\frac{dr}{dl}=0$ or $\frac{dR}{dg}=0$.

RECENT RESULTS IN THE STUDY OF LINKAGES.

BY PROF. W. W. JOHNSON, ANNAPOLIS, MD.

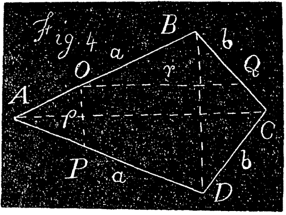
(*Continued from page* 46.)

THE so called kite-shaped quadrilateral, that which has two pairs of equal adjacent sides, may be employed as a cell in which the rays are always at right angles. Let $ABCD$ be such a quadrilateral the length of its unequal sides being a and b. Take any point on AB for the fulcrum O and draw OQ and OP parallel to the diagonals. It is evident that however the figure is deformed these lines are parallel to the diagonals which last are always at right angles. Denoting the diagonals AC and BD by x and y, and the segments of AC by z and v, we have

$$a^2 - b^2 = z^2 - v^2 = x(z - v),$$

and $2(a^2+b^2)=y^2+2z^2+2v^2=y^2+(z+v)^2+(z-v)^2$,

whence $y^2 + x^2 + \dfrac{(a^2-b^2)^2}{x^2} = 2(a^2 + b^2)$. . . (7)



Let $AO = ma$ then $\rho = my$, and $r = (1-m)x$; substituting in (7) we find

$$\rho^2 + \frac{m^2r^2}{(1-m)^2} + \frac{m^2(1-m)^2(a^2-b^2)^2}{r^2} = 2m^2(a^2 + b^2), \ldots\ldots (8)$$

a relation in which the constants may be any positive quantities since m, a and b are arbitrary.

By this cell the circle $\qquad \rho = c\cos\theta$

is transformed into

$$\frac{m^2}{(1-m)^2}(x^2+y^2)^2+(c^2-2m^2a^2-2m^2b^2)x^2-2(a^2+b^2)y^2+m^2(1-m)^2(a^2-b^2)^2=0,. (9)$$

a bicircular quartic symmetrical to both axes.

If O, P and Q be taken at the vertices of similar triangles similarly placed on AB, AD and CB, OP and OQ will make constant angles with the diagonals BD and AC, and consequently constant angles with one another. We shall therefore still have a cell in which the rays make a constant angle. The rays will still bear fixed ratios to the diagonals and therefore have a relation similar in form to equation (8).

An interesting application of Peaucellier's cell (see fig. 1, p. 41, ANALYST, Vol. II) made by Mr. G. H. Darwin in the mechanical transformation of a constant force into one which varies inversely as the square of the distance from a fixed point; by which he remarks it may be "possible to construct a toy to give an ocular proof of elliptic motion." Let the rays be r and ρ, then $r\rho = k^2$; and let F and φ be forces in equilibrium applied in the directions ρ and r, regarded as positive when they tend to increase these quantities. Then differentiating we have

$$d\rho = -\frac{k^2dr}{r^2},$$

but from the principle of virtual velocities,

$$Fd\rho = -\varphi dr,$$

hence $\qquad \varphi = k^2Fr^{-2}$, ... (10)

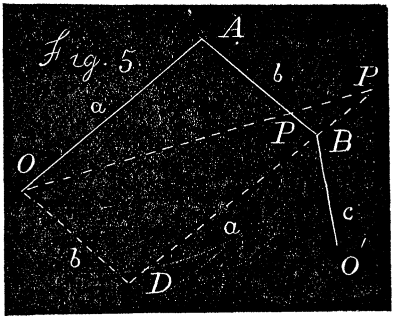
that is, a constant force F applied to the extremity of ρ and directed away from the fulcrum is equivalent to an attractive force $-\varphi$ applied at the extremity of r and varying inversely as the square of the distance from the fulcrum.

M. Mannheim, in a letter to Prof. Sylvester published in the proceeding of the London Mathematical Society, notices that, for the three bars on one side of the axis of a Peaucellier cell, we may substitute three other bars whose squares differ equally from the squares of the bars they replace; the effect of the change is simply to move the joint where the three bars meet in a line perpendicular to the axis. Replacing again the two bars which meet at the extremity of a ray by others whose squares differ equally from those of the bars they replace, the tracer fulcrum and follower are still in a straight line and the cell is equivalent to the Scalene cell. He then derives geometrically from this cell the results already alluded to as established analytically by Prof. Cayley.

Rotation about a fixed point may be regarded as the simplest variety of link motion, or 1-bar motion, the curves described being all circles. In a 3-bar linkwork, the result of fixing one side of a jointed quadrilateral, two of the bars have simple rotation while the third is said to have distinctively 3-bar motion. So in a 5-bar linkwork at least one bar will have 3-bar motion while one or more may have distinctively 5-bar motion. The curves described by points off the bars seem to be of the same character as those described by points on the bars, differing only in the values of the constants introduced.

The general curve in the case of the 3-bar motion is a sextic, but in two cases it breaks up into a circle and a quartic. The sextic is of course symmetrical to the line joining the pivots, and it has three nodes. When the nodes are all conjugate points, the curve may consist of two disconnected ovals, but generally, the curve may be described by continuous motion. When it breaks up into a circle and quartic the quartic retains one node, and the other two nodes of the sextic are represented by points of contact between the circle and quartic. The cases alluded to are those in which the quadrilateral formed by the bars and the line joining the pivots has two pairs of equal sides, either opposite or adjacent. When the opposite sides are equal the circle is described while the figure is in the form of a parallelogram. The quartic described in this case was first shown by Mr. Hart to be the inverse of a conic [see ANALYST for March, 1875].

In the case when the adjacent sides are equal, the circle is described when the equal sides of the figure are brought into coincidence, which permits the two equal bars to revolve about a pivot; and the quartic is described while the figure is in the form of a kite-shaped quadrilateral. M. Mannheim has given a geometrical proof that in this latter case the locus described by a point off the bar, but rigidly connected with it, is always the pedal of a conic, or what is the same thing the inverse of a conic.



Mr. A. B. Kempe pointed out this remarkable circumstance in connection with three-bar motion, viz; that the *shape* of the curves described depends solely upon the lengths of the moving bars, without reference to their order. Thus in Fig. 5, O and O' being the fixed pivots, for every curve described by a point on b, there exists a similar curve described by a point on a, when a is the middle bar. Prof. Cayley observed that when once stated this was an easy deduction from the principle of the pantagraph· Complete the parallelogram $OABD$, then $ODBO'$ is one of the positions of the linkage when the bars are in the order b, a, c. Take P' so that O, P and P' are in a straight line; then P and P' may be regarded as the tracer and follower in an ordinary pantagraph; accordingly P' describes a curve similar to that described by P. If $AP = mb$, the ratio of similitude is $1:m$, when $m = 0$, P is at A and describes a circle and P' is at infinity, which shows that the ultimate shape of the 3-bar curve when indefinitely increased is circular.

Sylvester states that it was this principle and its demonstration by the pantagraph which led him to the discovery of the skew-pantagraph; the corresponding curves which he found to exist also in the case of points off the bars being similar but not similarly placed, and derivable one from another by the linkage to which he has given the above name. The two cases in which the sextic breaks up into a circle and quartic are connected also by Kempe's principle, which has its application also to linkages of higher orders.

Mr. Kempe was the first to produce rectilinear motion by linkwork, otherwise than on the principle of inversion of the circle. This was effected by means of a 7-bar linkwork of which he published an account in the Mathematical Messenger for Dec., 1874. He has since developed a general principle from which he derives a large number of 7-bar rectilinear motions, of which the previously discovered ones including M. Peaucellier's and Mr. Hart's are special cases, "the inversion property of the last two being" to use his words "so to say, accidental." (See Proceedings of the Royal Society, No. 163, 1875.) His method depends upon the formation of a 6-bar linkage in which the projection upon one of the bars of the line joining two points of the linkage is constant. He first points out that the cosines of the opposite angles in a quadrilateral whose sides are given are connected by a linear relation. Let the sides of the quadrilateral $ABCD$, Fig. 6, be denoted by a, b, c and d, then equating two values of the square of the diagonal

AC, we have $a^2 + b^2 - 2ab\cos B = c^2 + d^2 - 2dc\cos D$. . . . . . (11) Now add to the 4-bar linkage *ABCD* the bars $B'C'$ and $C'D'$, in such a manner that $AB'C'D'$ shall be a quadrilateral similar to *ABCD*; then the angle $BD'C'$ is the supplement of *D*. Take *P* on *BC* and P' on $D'C$ such that $BP : D'P' = ab : cd$, then the projections *BN* and $D'N'$ will be proportional to $2ab\cos B$ and $-2dc\cos D$, and therefore in accordance with equation (11) we shall have $BN + N'D =$ a constant, hence also NN' is constant. Since *BP* is arbitrary any desired value may be given to NN', and since the ratio of similitude of the two quadrilaterals is arbitrary *B* and D' may be made to coincide.

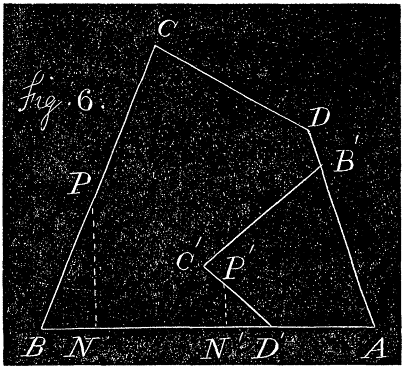


Fig. 6.

Among the various methods in which Mr. Kempe employs this fundamental linkage the following will be intelligible without reproducing the author's analytical work and diagrams.

First, fixing the point P' as a pivot and adding a bar which will cause *AB* to remain always parallel to a fixed direction (for instance, a bar *AO* equal and parallel to *BP*) *P* will move in the straight line *PN*.

Secondly, fixing the points *B*, D' and *A* (so that the bar *AB* may be removed), taking *BP* so that NN' is $\frac{1}{2}BD'$, and adding bars *PO*, OP' such that $PO = PB$, $P'O = P'D$, *O* will move in the straight line *AB*.

Thirdly, fixing the same points making D' coincide with *B*, and taking *BP* so that $NN' = 0$; we may make *O* move in the straight line *AB*, by taking the bars *PO*, $P'O$ so that $(P'O)^2 - (PO)^2 = (P'B)^2 - (PB)^2$; or we may make it move in a straight line perpendicular to *AB* at *B*, by taking the bars so that $PO = P'B$ and $P'O = PB$.

Fourthly, the same points being fixed and D' still coinciding with *B*, complete the parallelogram $P'BPR$, if the side *RP* be produced to *O* making $PO = PR$ the projection of the broken line *BPO* is evidently equal to that of the broken line $P'BP$, but this is constant, hence *O* moves in a perpendicular to *AB*.

Note.—That a revolving ellipsoid of three unequal axes can be in equilibrium is said to be proven in a paper by Mr. Ivory in the *Philosophical Transactions* for 1838, page 57. Cannot some of your subscribers inform us in regard to that paper? Walter Siverly.